Two families of particles with partial annihilation on a membrane

Wai-Tong (Louis) Fan
University of Washington

April 8, 2014

Department of Ecology and Evolutionary Biology
Princeton University

(Joint work with Zhen-Qing Chen)
Two families of particles with partial annihilation on a membrane
Two families of particles with partial annihilation on a membrane

- Interface $I$
- Harvest sites $\Lambda_{\pm}$
Two families of particles with partial annihilation on a membrane.

The image shows a diagram of two families of particles, labeled $\Lambda_+$ and $\Lambda_-$, interacting on a membrane. The particles are indicated by the symbols $+$ and $-$.
Two families of particles with partial annihilation on a membrane
$N = \text{initial number of particles}$

$\epsilon \approx N^{-1/d}$ i.e. $N\epsilon^d = O(1)$

per pair annihilation rate $\approx \lambda/\epsilon$
Can the particle system be a model for a biological or ecological system?
How will the system evolve?
e.g. How many charges are harvested? How many charges are annihilated?
Questions

- Can the particle system be a model for a biological or ecological system?
- How will the system evolve?
- e.g. How many charges are harvested? How many charges are annihilated?
Can the particle system be a model for a biological or ecological system?

How will the system evolve?

e.g. How many charges are harvested? How many charges are annihilated?
Questions

- Can the particle system be a model for a biological or ecological system?
- How will the system evolve?
  - e.g. How many charges are harvested? How many charges are annihilated?
Questions

- What is a good estimate for the number of particles in $K$?
- What is the prediction error?

Key: Scale of observation
What is a good estimate for the number of particles in $K$?
What is the prediction error?

Key: Scale of observation
What is a good estimate for the number of particles in $K$?  
What is the prediction error?

Key: Scale of observation
What is a good estimate for the number of particles in $K$?

What is the prediction error?

Key: Scale of observation
Assign a mass of $1/N$ to each particle.

**Theorem (Chen and Fan, 2013)**

Mass distributions $\rightarrow (u_+(t, x), u_-(t, y))$ satisfying

$$
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ \quad \text{on } D_+ \\
u_+ &= 0 \quad \text{on } \Lambda_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \quad \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \quad \text{on } I
\end{align*}
$$

and similar equations for $u_-$ in $D_-$. 

Two families of particles with partial annihilation on a membrane.
Assign a mass of \(1/N\) to each particle.

**Theorem (Chen and Fan, 2013)**

*Mass distributions* \(\rightarrow (u_+(t, x), u_-(t, y))\) satisfying

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ \quad \text{on } D_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \quad \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \quad \text{on } I \\
u_+ &= 0 \quad \text{on } \Lambda_+ \\
u_+ &= 0 \quad \text{on } \Lambda_-
\end{align*}
\]

and similar equations for \(u_-\) in \(D_-\).
Assign a mass of $1/N$ to each particle.

**Theorem (Chen and Fan, 2013)**

Mass distributions $\rightarrow (u_+(t, x), u_-(t, y))$ satisfying

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ \quad \text{on } D_+ \\
u_+ &= 0 \quad \text{on } \Lambda_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \quad \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \quad \text{on } I \\
\end{align*}
\]

and similar equations for $u_-$ in $D_-$. 

Two families of particles with partial annihilation on a membrane.
Macroscopic Limit

Assign a mass of $1/N$ to each particle.

Theorem (Chen and Fan, 2013)

*Mass distributions* $\to (u_+(t, x), u_-(t, y))$ satisfying

$$
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ & \text{on } D_+ \\
u_+ &= 0 & \text{on } \Lambda_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 & \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- & \text{on } I \\
\end{align*}
$$

and similar equations for $u_-$ in $D_-$. 

Two families of particles with partial annihilation on a membrane.
Macroscopic Limit

Theorem (Chen and Fan, 2013)

Mass distributions \( \rightarrow (u_+(t, x), u_-(t, y)) \)
satisfying

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= a \Delta u_+ + \vec{b} \cdot \nabla u_+ \quad \text{on } D_+ \\
u_+ &= 0 \quad \text{on } \Lambda_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \quad \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \quad \text{on } I
\end{align*}
\]

with similar equations for \( u_- \) in \( D_- \).

Applications:

- Landscape heterogeneity, external stressor
- Obtain an optimal drift for the solar cell

Two families of particles with partial annihilation on a membrane
Macroscopic Limit

Theorem (Chen and Fan, 2013)

Mass distributions $\rightarrow (u_+(t, x), u_-(t, y))$ satisfying

$$\begin{aligned}
\frac{\partial u_+}{\partial t} &= a \Delta u_+ + \vec{b} \cdot \nabla u_+ \\
u_+ &= 0 \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \\
\end{aligned}$$

on $D_+$

on $\Lambda_+$

on $\partial D_+ \setminus (\Lambda_+ \cup I)$

on $I$

with similar equations for $u_-$ in $D_-$. 

Applications:

- Landscape heterogeneity, external stressor
- Obtain an optimal drift for the solar cell
Macroscopic Limit

Theorem (Chen and Fan, 2013)

Mass distributions \( \rightarrow (u_+(t, x), u_-(t, y)) \)
satisfying

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= a \Delta u_+ + \vec{b} \cdot \nabla u_+ \quad \text{on } D_+ \\
\quad u_+ &= 0 \quad \text{on } \Lambda_+ \\
\frac{\partial u_+}{\partial \vec{n}_+} &= 0 \quad \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\
\frac{\partial u_+}{\partial \vec{n}_+} &= \lambda u_+ u_- \quad \text{on } I \\
\end{align*}
\]

with similar equations for \( u_- \) in \( D_- \).

Applications:

- Landscape heterogeneity, external stressor
- Obtain an optimal drift for the solar cell
Macroscopic Limit

Applications:

\[
\frac{\text{Number of particles in } K}{N} \approx \int_K u_+(t, x) \, dx
\]
Macroscopic Limit

Mass of particles in $K$

Number of particles in $K$ and $\int_K u_+(t, x) \, dx$

● How big is the error?
Number of particles in $K$ \( N \) and \( \int_\mathcal{K} u_+(t, x) \, dx \)

How big is the error?
Fluctuations

$Y_{t}^{N,+}(K) := \sqrt{N} \left( \frac{\text{Number of particles in } K}{N} - \int_{K} u_{+}(t, x) \, dx \right)$
Fluctuations

What is the probability distribution of $\mathcal{Y}_s^{N,\pm}(K)$ when $N$ is large?
Fluctuations

**Fluctuations in K**

- JOINT probability distribution of $\gamma_s^{N,+}(K)$ and $\gamma_t^{N,+}(K)$?
- What if we pick another region $K$?
Fluctuations

What if we pick another region $K$?

- Joint probability distribution of $\gamma^{N,+}_s(K)$ and $\gamma^{N,+}_t(K)$?
Consider

\[ Z_t^N(K^+, K^-) := \mathcal{Y}_t^{N,+}(K^+) + \mathcal{Y}_t^{N,-}(K^-) \]

**Theorem (Chen and Fan, 2014)**

\[ Z^N \rightarrow Z \] which solves

\[ dZ_t = A Z_t \, dt + dM_t, \]

where \( M \) is a Gaussian martingale with independent increments and variance

\[
\int_0^t \left( \int_{D_+} |\nabla \phi_+(x)|^2 u_+(s, x) \, dx + \int_{D_-} |\nabla \phi_-(y)|^2 u_-(s, y) \, dy \\
+ \int_I (\phi_+(z) + \phi_-(z))^2 u_+(s, z)u_-(s, z) \, d\sigma(z) \right) \, ds.
\]
Implications of the fluctuation theorem:

- \( \{ Z_t(K^+, K^-) : t \geq 0, K^+, K^- \} \) is a Gaussian system.

The curve \( t \mapsto Z_t(K^+, K^-) \) can be simulated by ‘transforming’ a Brownian motion.

- Noise comes from 2 correlated sources: diffusive transport and interactions.
Implications of the fluctuation theorem:

- \( \{ Z_t(K^+, K^-) : t \geq 0, K^+, K^- \} \) is a Gaussian system.

The curve \( t \mapsto Z_t(K^+, K^-) \) can be simulated by ‘transforming’ a Brownian motion.

- Noise comes from 2 correlated sources: diffusive transport and interactions.
Implications of the fluctuation theorem:

- $\{ Z_t(K^+, K^-) : t \geq 0, K^+, K^- \}$ is a Gaussian system.

- The curve $t \mapsto Z_t(K^+, K^-)$ can be simulated by 'transforming' a Brownian motion.

- Noise comes from 2 correlated sources: diffusive transport and interactions.
Related models

- Reaction diffusion systems
- Measure-valued processes (e.g. Super Brownian motion, **Fleming-Viot type processes**)
On going work


Wai-Tong (Louis) Fan  University of Washington  Two families of particles with partial annihilation on a membrane
Number of particles in $K$ in $N$ 

\[ \approx \int_K u_+(t, x) \, dx + \frac{\text{Gaussian}(K, t)}{\sqrt{N}} \]
• Multiple deletion (e.g. $u^2_+ u_-$)
• Multidimensional model
• Multi-type particles
• Anomalous diffusions
• Branching-Annihilation models


Two families of particles with partial annihilation on a membrane
Two families of particles with partial annihilation on a membrane
Two families of particles with partial annihilation on a membrane
Thank you!