Systems of reflected diffusions with annihilations through membranes

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Motivation and main results

Hydrodynamic limit

Fluctuation limit

Question 1

Let $X^i$ ($i = 1, 2, \cdots, N$) be independent RBMs in $D \subset \mathbb{R}^d$. Define

$$\mathcal{X}_t^N (dz) \triangleq \frac{1}{N} \sum_{i=1}^{N} 1_{\{X^i_t\}} (dz)$$

**Question:** Does $\mathcal{X}^N$ converge when $N \to \infty$?

**Answer:** LLN

If $\mathcal{X}_0^N \xrightarrow{L} f(x) \, dx$, then

$$\mathcal{X}_t^N \xrightarrow{L} u(t, x) \, dx$$

where $u$ is solution of Neumann heat equation, $u(0, \cdot) = f$. 

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Question 2

Fluctuation of $\mathcal{X}_t^N$ is

$$\gamma_t^N(\phi) := \sqrt{N} (\langle \mathcal{X}_t^N, \phi \rangle - \mathbb{E}\langle \mathcal{X}_t^N, \phi \rangle),$$

Question: Does $\gamma_t^N$ converge?
Question 2

Fluctuation of $\mathcal{X}_t^N$ is

$$\mathcal{Y}_t^N(\phi) := \sqrt{N} (\langle \mathcal{X}_t^N, \phi \rangle - \mathbb{E} \langle \mathcal{X}_t^N, \phi \rangle),$$

**Question:** Does $\mathcal{Y}^N$ converge?
Suppose initial positions of particles are i.i.d. Then $\mathcal{Y}^N_t \xrightarrow{L} \mathcal{Y}_t$, where

$$\mathcal{Y}_t = T_t \mathcal{Y}_0 + \int_0^t T_{t-s} dM_s, \quad \text{(Generalized O-U)}$$

where $M$ is a Gaussian martingale with independent increments and known covariance, and $T_t \mu(\phi) \triangleq \mu(\mathcal{P}_t \phi)$.

Example:

$$\mathcal{Y}_t(\phi) \overset{L}{=} \mathcal{Y}_0(\mathcal{P}_t \phi) + \int_0^t \sqrt{\iint_D \left| \nabla \mathcal{P}_t \phi(x) \right|^2 u(s, x) \, dx \, dB_s}.$$
Suppose initial positions of particles are i.i.d. Then $\mathcal{Y}_t^N \overset{L}{\to} \mathcal{Y}_t$, where

$$
\mathcal{Y}_t = T_t \mathcal{Y}_0 + \int_0^t T_{t-s} \, dM_s, \quad \text{(Generalized O-U)}
$$

where $M$ is a Gaussian martingale with independent increments and known covariance, and $T_t \mu(\phi) \triangleq \mu(P_t \phi)$.

Example:

$$
\mathcal{Y}_t(\phi) \overset{L}{=} \mathcal{Y}_0(P_t \phi) + \int_0^t \sqrt{\int_D |\nabla P_{t-s} \phi(x)|^2 \, u(s, x) \, dx \, dB_s}.
$$
Answer: CLT / Fluctuation-Dissipation Theorem

Suppose initial positions of particles are i.i.d. Then $\mathcal{Y}_t^N \xrightarrow{L} \mathcal{Y}_t$, where

$$\mathcal{Y}_t = T_t \mathcal{Y}_0 + \int_0^t T_{t-s} dM_s, \quad \text{(Generalized O-U)}$$

where $M$ is a Gaussian martingale with independent increments and known covariance, and $T_t \mu(\phi) \triangleq \mu(P_t \phi)$.

Example:

$$\mathcal{Y}_t(\phi) \xleftarrow{L} \mathcal{Y}_0(P_t \phi) + \int_0^t \sqrt{\int_D \left| \nabla P_{t-s} \phi(x) \right|^2 u(s, x) \, dx \, dB_s}.$$
Motivation

- Interface \( I \)
- Harvest sites \( \Lambda_{\pm} \)
Motivation

Hydrodynamic limit

Fluctuation limit

Motivation

\[ dX_t^\pm = dB_t^\pm + \frac{1}{2} \nabla (\log \rho_\pm (X_t^\pm)) \, dt + \bar{n}(X_t^\pm) \, dL_t \]
Motivation

- $N =$ initial number of particles
- annihilation distance $\delta_N \approx N^{-1/d}$,
- per pair annihilation rate $\approx 1/\delta_N$
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Motivation

\[ N = \text{initial number of particles} \]
\[ \delta_N \approx N^{-1/d}, \]
\[ \text{per pair annihilation rate} \approx \frac{1}{\delta_N} \]
Empirical measures

Define

\[
\mathcal{X}_t^{N,+}(dx) \triangleq \frac{1}{N} \sum_{\alpha \sim t} 1_{x_{\alpha}^+(t)}(dx) \quad \text{on } \overline{D}_+
\]

\[
\mathcal{X}_t^{N,-}(dy) \triangleq \frac{1}{N} \sum_{\beta \sim t} 1_{x_{\beta}^-(t)}(dy) \quad \text{on } \overline{D}_-
\]

What does \((\mathcal{X}_t^{N,+}, \mathcal{X}_t^{N,-})\) look like when \(N \to \infty\)?
Empirical measures

Define

\[
\mathcal{X}^{N,+}_t(dx) \triangleq \frac{1}{N} \sum_{\alpha \sim t} 1_{X^+_{\alpha}(t)}(dx) \quad \text{on} \quad \overline{D}_+
\]

\[
\mathcal{X}^{N,-}_t(dy) \triangleq \frac{1}{N} \sum_{\beta \sim t} 1_{X^-_{\beta}(t)}(dy) \quad \text{on} \quad \overline{D}_-
\]

What does \((\mathcal{X}^{N,+}_t, \mathcal{X}^{N,-}_t)\) look like when \(N \to \infty\)?
Main results

Theorem (Chen, F, 2013)

(\textbf{LLN}, No drift case) If \((X_0^N, Y_0^N)\) converges, then

\[
(X_t^{N,+}, X_t^{N,-}) \xrightarrow{L} (u_+(t, x) \, dx, \, u_-(t, y) \, dy),
\]

where \((u_+, u_-)\) is the solution to

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ & \text{on } (0, \infty) \times D_+ \\
\frac{\partial u_+}{\partial n_+} &= u_+ u_- 1_\Lambda & \text{on } (0, \infty) \times (\partial D_+ \setminus \Lambda_+)
\end{align*}
\]

with similar equations for \(u_-\) in \(D_-\).
Main results

**Theorem (Chen, F, 2013)**

*(LLN, No drift case)* If \((X_0^N, Y_0^N)\) converges, then

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\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ \quad \text{on} \ (0, \infty) \times D_+ \\
\frac{\partial u_+}{\partial n_+} &= u_+ u_- 1_I \quad \text{on} \ (0, \infty) \times (\partial D_+ \setminus \Lambda_+) \\
\end{align*}
\]

with similar equations for \(u_-\) in \(D_-\).
Main results

Theorem (Chen, F, 2013)

(\textbf{LLN}) If \((X^N_0, Y^N_0)\) converges, then

\[ (X^{N,+}_t, X^{N,-}_t) \xrightarrow{L} (u_+(t, x) \rho_+(x) dx, u_-(t, y) \rho_-(y) dy), \]

where \((u_+, u_-)\) is the solution to

\[
\begin{align*}
\frac{\partial u_+}{\partial t} &= \frac{1}{2} \Delta u_+ + \frac{1}{2} \nabla (\log \rho_+) \cdot \nabla u_+ & \text{on } (0, \infty) \times D_+ \\
u_+ &= 0 & \text{on } (0, \infty) \times \Lambda_+ \\
\frac{\partial u_+}{\partial n_+} &= \frac{1}{\rho_+} u_+ u_- 1_I & \text{on } (0, \infty) \times (\partial D_+ \setminus \Lambda_+) \text{ with similar equations for } u_- \text{ in } D_-.
\end{align*}
\]
Theorem (Chen, F, 2013) \textit{(Propagation of chaos)} Pick $n$ and $m$ particles in $D_+$ and $D_-$. The joint probability distribution for their positions converges to

$$\prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j)$$

uniformly on compacts, up to a normalizing constant.
Main results

Theorem (Chen, F, 2013)

(Propagation of chaos) Pick $n$ and $m$ particles in $D_+$ and $D_-$. The joint probability distribution for their positions converges to

$$ \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) $$

uniformly on compacts, up to a normalizing constant.
Fluctuations

\[ \mathcal{V}_t^{N,\pm}(\phi) := \sqrt{N} \left( \langle X_t^{N,\pm}, \phi \rangle - \mathbb{E}\langle X_t^{N,\pm}, \phi \rangle \right), \]

Consider \( Z_t^N := \mathcal{V}_t^{N,+} \oplus \mathcal{V}_t^{N,-} \) defined as

\[ Z_t^N(\phi_+, \phi_-) := \mathcal{V}_t^{N,+}(\phi_+) + \mathcal{V}_t^{N,-}(\phi_-) \]
Fluctuations

\[ \mathcal{Y}^{N,\pm}_t(\phi) := \sqrt{N} \left( \langle X_t^{N,\pm}, \phi \rangle - E\langle X_t^{N,\pm}, \phi \rangle \right), \]

Consider \( \mathcal{Z}^N_t := \mathcal{Y}^{N,+}_t \oplus \mathcal{Y}^{N,-}_t \) defined as

\[ \mathcal{Z}^N_t(\phi_+, \phi_-) := \mathcal{Y}^{N,+}_t(\phi_+) + \mathcal{Y}^{N,-}_t(\phi_-) \]
Theorem (Chen, F, 2014) \((\text{CLT})\) Suppose initial positions of particles in \(\overline{D}_\pm\) are i.i.d. Then \(Z^N \xrightarrow{L} Z\) in \(D([0, T_0], H)\), where

\[
Z_t = U_{(t,0)} Z_0 + \int_0^t U_{(t,s)} \, dM_s.
\]

\(M\) is a \(H\)-valued Gaussian martingale with independent increments and covariance

\[
\langle M(\phi_+, \phi_-) \rangle_t = \int_0^t \left( \int_{D_+} |\nabla \phi_+|^2 u_+(s) + \int_{D_-} |\nabla \phi_-|^2 u_-(s) \right. \\
\left. + \int_I (\phi_+ + \phi_-)^2 u_+(s)u_-(s) \, d\sigma \right) \, ds.
\]
Theorem (Chen, F, 2014) \((\text{CLT})\) Suppose initial positions of particles in \(\overline{D}_\pm\) are i.i.d. Then \(Z^N \xrightarrow{L} Z\) in \(D([0, T_0], H)\), where

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\langle M(\phi_+, \phi_-) \rangle_t = \int_0^t \left( \int_{D_+} |\nabla \phi_+|^2 u_+(s) + \int_{D_-} |\nabla \phi_-|^2 u_-(s) + \int_I (\phi_+ + \phi_-)^2 u_+(s) u_-(s) \, d\sigma \right) \, ds.
\]
Motivation and main results

Hydrodynamic limit

Fluctuation limit

Main results

Corollary

(Properties of $M$ and $Z$)

\[ M \overset{d}{=} M^+ \oplus M^- , \]

where $M^\pm$ is a continuous $\mathcal{H}^\pm$-valued Gaussian martingale with known covariance and

\[
\mathbb{E} [ M^+_s (\phi_+) M^-_t (\psi_-) ] = \int_0^{s \land t} \int_I \phi_+ \psi_- u_+(r) u_- (r) d\sigma dr
\]

$Z$ is a continuous Gaussian Markov process.
Main results

Corollary

(Properties of $M$ and $Z$)

\[ M \overset{c}{=} M^+ \oplus M^-, \]

where $M^\pm$ is a continuous $\mathcal{H}^\pm$-valued Gaussian martingale with known covariance and

\[ \mathbb{E}[M_s^+ (\phi_+) M_t^- (\psi_-)] = \int_0^{s \wedge t} \int_I \phi_+ \psi_- u_+ (r) u_- (r) \, d\sigma \, dr \]

$Z$ is a continuous Gaussian Markov process.
Fleming-Viot type systems


... Reaction diffusion systems $\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + R(u)$

- C. Boldrighini, A. De Masi, A. Pellegrinotti, etc (1986, 1987)
- P. Kotelenez (1982-88)
- P. Dittrich (1988)

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Motivation and main results
Hydrodynamic limit
Fluctuation limit

Fluctuations

- P. Kotelenez and P. Dittrich (1986, 1988)
- C. Boldrighini, A. De Masi, A. Pellegrinotti (1992)

\[ \ldots \]
Motivation and main results
Hydrodynamic limit
Fluctuation limit

Discrete model: Description

- $D^\epsilon_{\pm} \triangleq D_{\pm} \cap \epsilon \mathbb{Z}^d$, where $\epsilon = O(N^{-1/d})$.
- Per pair annihilation rate: $1/\epsilon$
Motivation and main results
Hydrodynamic limit
Fluctuation limit

Discrete model: Description

- Split $I$ into pieces of size $O(\epsilon^{(d-1)})$
- Pick a pair $(z_+, z_-)$ for each piece and record it in $I^\epsilon$ (multiplicity counted)
Discrete model: Construction

- \( \eta_t^{\epsilon, +}(x) \): the number of "+" particles at \( x \in D_+^{\epsilon} \)
- \( \eta_t^{\epsilon, -}(x) \): the number of "-" particles at \( x \in D_-^{\epsilon} \)
- A configuration at time \( t \) is

\[
\eta_t^{\epsilon} = (\eta_t^{\epsilon, +}, \eta_t^{\epsilon, -}) \in \mathbb{N}^{D_+^{\epsilon}} \times \mathbb{N}^{D_-^{\epsilon}}
\]
Discrete model: Construction

- $\eta_{t}^{\epsilon, +}(x)$: the number of "+" particles at $x \in D_{+}^{\epsilon}$
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A configuration at time $t$ is

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Discrete model: Construction

- $\eta^\epsilon,^+ (x)$: the number of "+" particles at $x \in D_+^\epsilon$
- $\eta^\epsilon,^- (x)$: the number of "－" particles at $x \in D_-^\epsilon$
- A configuration at time $t$ is

$$\eta^\epsilon_t = (\eta^\epsilon,^+, \eta^\epsilon,^-) \in \mathbb{N}^{D_+^\epsilon} \times \mathbb{N}^{D_-^\epsilon}$$
Generator for $\eta_t^\epsilon$ is $\mathcal{L}_0^\epsilon + \mathcal{R}^\epsilon$, where

$$\mathcal{L}_0^\epsilon f(\eta) := \frac{d}{\epsilon^2} \sum_{x, y \in D_+^\epsilon} \eta^+(x) \rho_{xy}^+ \{ f(\eta^+ - 1_x + 1_y, \eta^-) - f(\eta) \}$$

$$+ \frac{d}{\epsilon^2} \sum_{x, y \in D_-^\epsilon} \eta^-(x) \rho_{xy}^- \{ f(\eta^+, \eta^- - 1_x + 1_y) - f(\eta) \} ,$$

$$\mathcal{R}^\epsilon f(\eta) := \frac{1}{\epsilon} \sum_{(z_+, z_-) \in I^\epsilon} \Psi_\epsilon(z) \eta^+(z_+) \eta^-(z_-) \{ f(\eta^+ - 1_{z_+}, \eta^- - 1_{z_-}) - f(\eta) \} .$$
Generator for $\eta^t_\epsilon$ is $\mathcal{L}_0^\epsilon + \mathcal{K}^\epsilon$, where

$$\mathcal{L}_0^\epsilon f(\eta) := \frac{d}{\epsilon^2} \sum_{x,y \in D^\epsilon_+} \eta^+(x) \rho^+_{xy} \{ f(\eta^+ - 1_x + 1_y, \eta-) - f(\eta) \}$$

$$+ \frac{d}{\epsilon^2} \sum_{x,y \in D^\epsilon_-} \eta^-(x) \rho^-_{xy} \{ f(\eta^+ , \eta^- - 1_x + 1_y) - f(\eta) \},$$

$$\mathcal{K}^\epsilon f(\eta) := \frac{1}{\epsilon} \sum_{(z_+,z_-) \in I^\epsilon} \Psi^\epsilon(z) \eta^+(z_+) \eta^-(z_-) \{ f(\eta^+ - 1_{z_+}, \eta^- - 1_{z_-}) - f(\eta) \}.$$
Define the empirical measures

\[ \mathcal{X}_t^{N,+} (dz) \triangleq \frac{1}{N} \sum_{x \in D_+^e} \eta_{t}^+(x) \mathbf{1}_x (dz) \]

and

\[ \mathcal{X}_t^{N,-} (dz) \triangleq \frac{1}{N} \sum_{x \in D_-^e} \eta_{t}^-(x) \mathbf{1}_x (dz) \]

\( (\text{LLN}) \quad (\mathcal{X}_t^{N,+}, \mathcal{X}_t^{N,-}) \xrightarrow{L} (u_+(t, x) \, dx, \ u_-(t, y) \, dy). \)
Define the empirical measures

\[ \mathcal{X}^{N,+}_t (dz) \triangleq \frac{1}{N} \sum_{x \in D^+_+} \eta^+_t (x) \mathbf{1}_x (dz) \]

and

\[ \mathcal{X}^{N,-}_t (dz) \triangleq \frac{1}{N} \sum_{x \in D^-_-} \eta^-_t (x) \mathbf{1}_x (dz) \]

\((\text{LLN})\) \( (\mathcal{X}^{N,+}_t, \mathcal{X}^{N,-}_t) \xrightarrow{L} (u_+(t, x) \, dx, \, u_-(t, y) \, dy) \).
Motivation and main results
Hydrodynamic limit
Fluctuation limit

Idoof for LLN

\[ \gamma_t^{N,(n,m)} \rightarrow \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) \]

Tightness

Identify Limit

LLN

\[ (\mathcal{X}^{N,+}, \mathcal{X}^{N,-}) \rightarrow (u_+, u_-) \]
Idoof for LLN

\[ \gamma_t^{N,(n,m)} \rightarrow \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) \]

- Local CLT
- Uniqueness of BBGKY
- Duality
- Propagation of Chaos
- Tightness
- Identify Limit

\[ (X_t^{N,+}, X_t^{N,-}) \rightarrow (u_+, u_-) \]
Theorem (Burdzy, Chen, 2008)

Suppose $D$ is a $W^{1,2}$-extension domain.

Simple Random Walk on $D^\epsilon \xrightarrow{L} \text{RBM}$
Local CLT for RBM

Theorem (Chen, F, 2013) \( \text{(Local CLT)} \) Suppose \( D \) is bounded Lipschitz.

\[
\lim_{\epsilon \to 0} \sup_{t \in [a,b]} \sup_{x, y \in \overline{D}} \left| p^\epsilon(t, x, y) - p(t, x, y) \right| = 0
\]

Key: Establish "discrete relative isoperimetric inequality"
Theorem (Chen, F, 2013) \textbf{(Local CLT)} \textit{Suppose $D$ is bounded Lipschitz.}

$$\lim_{\epsilon \to 0} \sup_{t \in [a, b]} \sup_{x, y \in \overline{D}} |p^\epsilon(t, x, y) - p(t, x, y)| = 0$$

Key: Establish "discrete relative isoperimetric inequality"
Motivation and main results
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Coupled PDE

Theorem

The coupled PDE has a unique weak solution given by

\[
\begin{align*}
    u_+(t, x) &= \mathbb{E}^x \left[ f(X_t^+) \exp \left( - \int_0^t u_-(t - s, X_s^+) \, dL_s^+ \right) \right] \\
    u_-(t, y) &= \mathbb{E}^y \left[ g(X_t^-) \exp \left( - \int_0^t u_+(t - s, X_s^-) \, dL_s^- \right) \right]
\end{align*}
\]
Coupled PDE

\((u_+, u_-)\) satisfies the coupled integral equation

\[
\begin{align*}
u_+(t, x) &= P^+_t f(x) - \frac{1}{2} \int_0^t \int_I p^+(t - s, x, \cdot) u_+(s) u_-(s) \, d\sigma \, ds \\
u_-(t, y) &= P^-_t g(y) - \frac{1}{2} \int_0^t \int_I p^-(t - s, y, \cdot) u_+(s) u_-(s) \, d\sigma \, ds.
\end{align*}
\]

Iterating, we have

\[
\gamma^{(n,m)}(\vec{x}, \vec{y}) := \prod_{i=1}^n u_+(t, x_i) \prod_{j=1}^m u_-(t, y_j)
\]

solves a BBGKY hierarchy.
Motivation and main results
Hydrodynamic limit
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Coupled PDE

\((u_+ , u_-)\) satisfies the coupled integral equation

\[
\begin{align*}
 u_+(t, x) & = \mathbb{P}_t^+ f(x) - \frac{1}{2} \int_0^t \int_I p^+(t - s, x, \cdot) u_+(s) u_-(s) \, d\sigma \, ds \\
 u_-(t, y) & = \mathbb{P}_t^- g(y) - \frac{1}{2} \int_0^t \int_I p^-(t - s, y, \cdot) u_+(s) u_-(s) \, d\sigma \, ds.
\end{align*}
\]

Iterating, we have

\[
\gamma^{(n,m)}(\vec{x}, \vec{y}) := \prod_{i=1}^n u_+(t, x_i) \prod_{j=1}^m u_-(t, y_j)
\]

solves a BBGKY hierarchy.
Motivation and main results

Hydrodynamic limit

Fluctuation limit

BBGKY hierarchy

\[ \gamma_t^{(n,m)} = P_t^{(n,m)} \gamma_0^{(n,m)} - \int_{s=0}^{t} P_{t-s}^{(n,m)} \left( \sum_{i=1}^{n} V_i^+ \gamma_s^{(n,m+1)} + \sum_{j=1}^{m} V_j^- \gamma_s^{(n+1,m)} \right), \]

where

\[ V_i^+ \gamma^{(n,m+1)} := \gamma^{(n,m+1)}(\bar{a}, (\bar{b}, a_i)) \left. d\sigma \right|_{I(a_i)} d(\bar{a} \setminus a_i) d\bar{b}. \]
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**BBGKY hierarchy**

\( \gamma_t^{(n,m)} = P_t^{(n,m)} \gamma_0^{(n,m)} - \int_{s=0}^{t} P_{t-s}^{(n,m)} \left( \sum_{i=1}^{n} V_i^+ \gamma_s^{(n,m+1)} + \sum_{j=1}^{m} V_j^- \gamma_s^{(n+1,m)} \right) \),

where

\( V_i^+ \gamma^{(n,m+1)} := \gamma^{(n,m+1)}(\vec{a}, (\vec{b}, a_i)) \ d\sigma \bigg|_{I(a_i)} d(\vec{a} \setminus a_i) d\vec{b} \).
BBGKY hierarchy

\[
\gamma^{(n,m)}_t = P^{(n,m)}_t \gamma^{(n,m)}_0 - \int_{s=0}^t P^{(n,m)}_{t-s} \left( \sum_{i=1}^{n} V^+_i \gamma^{(n,m+1)}_s + \sum_{j=1}^{m} V^-_j \gamma^{(n+1,m)}_s \right),
\]

where

\[
V^+_i \gamma^{(n,m+1)} := \gamma^{(n,m+1)}(\vec{a}, (\vec{b}, a_i)) \, d\sigma \bigg|_{l}(a_i) \, d(\vec{a} \setminus a_i) \, d\vec{b}.
\]
Summary for LLN

- Local CLT
- Propagation of Chaos
  \[ \gamma_t^{N,(n,m)} \rightarrow \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) \]
- Tightness
- Identify Limit
- LLN
  \[ (X^{N,+}, X^{N,-}) \rightarrow (u_+, u_-) \]

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Hydrodynamic limit
Fluctuation limit

Systems of reflected diffusions with annihilations through membranes
Continuous model: construction

- \( I^\delta \triangleq \{(x, y) : \sqrt{|x - z|^2 + |y - z|^2} < \delta \text{ for some } z \in I\} \)
- Minkowski content:
  \[
  \lim_{\delta \to 0} \frac{\mathcal{H}_{2d}(I^\delta)}{C_{d+1} \delta^{d+1}} = \mathcal{H}_{d-1}(I)
  \]
Continuous model: construction

$I^\delta \triangleq \{(x, y) : \sqrt{|x - z|^2 + |y - z|^2} < \delta \text{ for some } z \in I\}$

Minkowski content:

$$\lim_{\delta \to 0} \frac{\mathcal{H}_{2d}(I^\delta)}{c_{d+1} \delta^{d+1}} = \mathcal{H}_{d-1}(I)$$
Motivation and main results

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Continuous model: Idoof for CLT

Idoof for CLT:

Recall that $\mathcal{Z}_t^N := \mathcal{Y}_t^{N,+} \oplus \mathcal{Y}_t^{N,-}$.

$$\mathcal{Z}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Z}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N dM_s^N + \text{Error}^N(t)$$

Asymptotic expansion of the $(n, m)$-correlation functions

$$\gamma_{t}^{N,(n,m)}(\vec{x}, \vec{y}) \approx \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) + \frac{G_{t}^{N,(n,m)}(\vec{x}, \vec{y})}{N} + \frac{o(N)}{N}$$

Boltzman-Gibbs principle

Wai-Tong (Louis) Fan
University of Washington

Systems of reflected diffusions with annihilations through membranes
Continuous model: Idoof for CLT

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- Asymptotic expansion of the $(n, m)$-correlation functions

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- Boltzmann-Gibbs principle
Motivation and main results

Hydrodynamic limit

Fluctuation limit

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Applications / Remarks

- From macro to micro: Control problem
- From micro to macro: Stochastic models for coupled PDE and SPDE
Applications / Remarks

- From macro to micro: Control problem
- From micro to macro: Stochastic models for coupled PDE and SPDE
On-going work / Open problems

- Large deviation principle ?
- Generalized O-U processes ?
- Branching models ?
- Hard annihilation ?
- Multi-types ? Multiple deletions ?
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Thank you!